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SPECTRAL GALERKIN METHOD FOR SOLVING HELMHOLTZ AND LAPLACE DIRICHLET PROBLEMS ON MULTIPLE OPEN ARCS

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Abstract. We present a spectral numerical scheme for solving Helmholtz and Laplace problems with Dirichlet boundary conditions on an unbounded non-Lipschitz domain $\mathbb{R}^2\backslash\overline{\Gamma}$, where Γ is a finite collection of open arcs. Through an indirect method, a first kind formulation is derived whose variational form is discretized using weighted Chebyshev polynomials. We show that our discretization basis allows for exponential convergence under smoothness assumptions. We show how a simple preconditioner can be built with successful results and introduce an efficient compression algorithm.

1 INTRODUCTION

We seek solutions of Helmholtz and Laplace equations in a two-dimensional plane after removing a finite collection of open finite curves –also called arcs. Applications for this problem can be found in various areas such as structural and mechanical engineering [14, 15, 2, 3, 8]; antenna design and acoustic engineering [13, 18]; and, biomedical imaging [1, 17]. Such problems pose the following challenges: (i) *non-Lipschitz domains*, for which weak formulations do not follow directly from Green formulae; (ii) *unbounded domains*, which call for boundary integral methods with carefully chosen radiation conditions; (iii) *singular behaviors* of solutions near arcs' end points; and, (iv) *large number of degrees of freedom* when the frequency or number of arcs increase.

Theoretical considerations concerning the first two points are well addressed in various contexts, in particular, we follow the approach presented in [5, 6]. Here, we show that we can extend the results form $\widehat{\Gamma} := (-1, 1) \times 0$ to more general arcs. The last two points deal with computational hindrances that can be addressed by employing spectral bases –specifically weighted Chebyshev polynomials– incorporating singular behaviors near the end points. The main objective of this work is prove that using this spectral basis we obtain an efficient numerical method capable of dealing with large numbers of curves and wavenumbers.

Yet, one more challenge remains and it concerns how well we can solve the linear system resulting of the discretization of the underlying boundary integral equations. Here, we show that is possible to obtain an approximation for a second kind formulation with good results and which enhanced by using the techniques described in [9, 4].

2 Problem Model

2.1 Notation

In what follows, the extended and non-negative integer values are denoted by $\mathbb{N}^* := \mathbb{N} \cup \{\infty\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, respectively. Vectors are indicated by boldface symbols. Euclidean norm is written $\|\cdot\|_2$ with other norms indicated by subscripts. For $k \in \mathbb{N}_0$ and a non-empty set G, $\mathcal{C}^k(G)$, denotes the set of continuous functions along with their k derivatives over G. Compactly supported $\mathcal{C}^k(G)$-functions are denoted by $\mathcal{C}^k_0(G)$. The class of all analytic functions will be denoted by $\mathcal{C}^\infty(G)$. Duality pairings are written as $\langle \cdot, \cdot \rangle$ with subscripts indicating the domain of involved functional spaces, if it is not clear from the context. Similarly, inner products are written as $(\cdot, \cdot)$, only requiring integration domains as subscript.

2.2 Geometry

In order to describe the domain where the problem is defined we need to give a precise description of what we understand for an open arc. We say that $\Lambda \subset \mathbb{R}^2$ is a regular Jordan arc (that will be refered just as arc) of class $\mathcal{C}^k$, $k \in \mathbb{N}^*$, if there exists a bijective $\mathcal{C}^k$-parametrization denoted by $\mathbf{r} = (r_1, r_2)$, such that $\mathbf{r} : [-1,1] \to \overline{\Lambda}$; $\mathbf{r} : (-1,1) \to \Lambda$, and $\|\mathbf{r}'(t)\|_2 > 0$, $\forall\, t \in [-1,1]$. We will assume that we can complete any open arc by a closed curve that keeps the same regularity.

Assumption 1. *For any Λ regular Jordan arc of class $\mathcal{C}^k$, there exists an extension of Λ to $\tilde{\Lambda}$, with a $\mathcal{C}^k$-parametrization $\tilde{\mathbf{r}} : [0, 2\pi] \to \tilde{\Lambda}$, that is bijective in $[0, 2\pi)$ and satisfies $\tilde{\mathbf{r}}(0) = \tilde{\mathbf{r}}(2\pi)$ and $\|\tilde{\mathbf{r}}'(t)\|_2 > 0$, for all $t \in [0, 2\pi]$.*

For a finite integer M, consider a collection of arcs $\{\Gamma_j\}_{j=1}^M$ each one of class $\mathcal{C}^k$, for $k \geq 2$. Denote by $\Gamma := \bigcup_{i=1}^M \Gamma_i$. For $m \in \mathbb{N}^*$, we claim that Γ is of class $\mathcal{C}^m$, if the parametrizations $r_i : [-1,1] \to \mathbb{R}^2, i \in \{1, \ldots, M\}$ are component-wise of class $\mathcal{C}^m((-1,1))$, where $m = \infty$ is the analytic case.

We define our problem domain as

$$\Omega := \mathbb{R}^2 \setminus \overline{\Gamma}. \tag{1}$$

We say a function $g : [-1,1] \to \mathbb{C}$ is analytic, if there exists a Bernstein ellipse of parameter $\rho > 1$, such that g is analytic –in the complex variable context– in the ellipse (*cf.* [16, Chapter 8]). For $\mathbf{g} = (g_1, \ldots, g_M)$ such that $g_j : \overline{\Gamma}_j \to \mathbb{C}$, for $j \in \{1, \ldots, M\}$, we say that $\mathbf{g}$ is of class $\mathcal{C}^m(\Gamma)$, if $g_i \circ r_i \in \mathcal{C}^m([-1,1])$, for $i \in \{1 \ldots M\}$, and denote $\mathbf{g} \in \mathcal{C}^m(\Gamma)$.

For each Γ_j we denote by Ω_j a bounded domain whose boundary contains Γ_j –by Assumption 1, there is at least one of this domains for every j– and we assume than we can select the collection $\{\Omega_j\}_{j=1}^M$ as disjoint domains.

Assumption 2. *The domains* $\{\Omega_i\}_{i=1}^M$ *are disjoint.*

2.3 Functional spaces

Let $G \subseteq \mathbb{R}^d$, $d = 1, 2$, be an open domain. For $s \in \mathbb{R}$, we denote by $H^s(G)$ the standard Sobolev spaces ([12, Section 2.3]) and $H^s_{loc}(G)$, their local integrable counterpart. As in [5, Section 2.3], for any Lipschitz open arc Λ that can be extended to a closed curve $\tilde{\Lambda}$, we define tilde spaces $\widetilde{H}^s(\Lambda)$ as

$$\widetilde{H}^s(\Lambda) := \{u \in \mathcal{D}'(\Lambda) : \tilde{u} \in H^s(\tilde{\Lambda})\}, \quad s > 0, \tag{2}$$

where $\tilde{u}$ denotes the extension by zero of u to $\widetilde{\Gamma}$. For $s > 0$ we can identify

$$\widetilde{H}^{-s}(\Lambda) = (H^s(\Lambda))^*, \quad \text{and} \quad H^{-s}(\Lambda) = (\widetilde{H}^s(\Lambda))^*. \tag{3}$$

We will also need the family mean-zero Sobolev spaces:

$$\widetilde{H}^s_{\langle 0\rangle}(\Lambda) = \{u \in \widetilde{H}^s(\Lambda) : \langle u, 1\rangle = 0\}. \tag{4}$$

For the finite union of disjoint open arcs Γ, as in Section 2.2, we define the piecewise spaces as

$$\mathbb{H}^s(\Gamma) := \left\{u \in \mathcal{D}^*(\Gamma) : u|_{\Gamma_i} \in H^s(\Gamma_i), i = 1 \dots M\right\}. \tag{5}$$

From this definition, the identification $\mathbb{H}^s(\Gamma) \cong H^s(\Gamma_1) \times \cdots \times H^s(\Gamma_M)$ follows. The norm and dual products are naturally extended by the previous identification. The spaces $\widetilde{\mathbb{H}}^s(\Gamma)$, and $\widetilde{\mathbb{H}}^s_{\langle 0\rangle}(\Gamma)$, are defined in similar fashion, imposing their components to be in $\widetilde{H}^s(\Gamma_j)$ and $\widetilde{H}^s_{\langle 0\rangle}(\Gamma_j)$, for $j = 1, \dots, M$, respectively.

For unbounded domains it is customary to use local Sobolev spaces, but since they are not Hilbert spaces, we rather use weighted spaces as in [5, Section 2.5][1]

$$W(G) := \left\{u \in \mathcal{D}^*(G) : \frac{u(\mathbf{x})}{\sqrt{1 + \|\mathbf{x}\|_2^2}\log(2 + \|\mathbf{x}\|_2^2)} \in L^2(G), \nabla u \in L^2(G)\right\}. \tag{6}$$

In [11] we show the inclusion $W(G) \subset H^1_{loc}(G)$, which allows to define trace operators on the space $W(G)$. In particular, for a family of arcs $\{\Gamma_i\}_{i=1}^M$, the Dirichlet traces operator $\gamma_i^\pm$ over Γ_i, $i \in \{1, \dots, M\}$, are defined over continuous functions u as

$$\gamma_i^\pm u(\mathbf{x}) := \lim_{\epsilon \uparrow 0} u(\mathbf{x} \pm \epsilon\mathbf{n}), \quad \forall\, \mathbf{x} \in \Gamma_i, \tag{7}$$

where $\mathbf{n}$ denote the unitary normal vector with direction of $(r_2', -r_1')$. The definition is extended to more general spaces by density arguments. If both traces are equal we simple write $\gamma_i u$.

[1]For the sake of clarity, we have dropped sup-indices $W^{1,-1}$ used in the original work.

2.4 Problem formulation

Now we can write the problem that we are interesting to study.

Problem 1. *Let $g \in \mathbb{H}^{\frac{1}{2}}(\Gamma)$ and the wavenumber k be real and non-negative. We seek $u \in H^1_{loc}(\Omega)$ such that*

$$\begin{aligned} -\Delta u - k^2 u &= 0 && \text{in}\, \Omega, && (8)\\ \gamma_i^{\pm} u &= g|_{\Gamma_i} && i = 1, \ldots, M, && (9)\\ &\textit{condition at infinity}(k). &&&& (10) \end{aligned}$$

Condition (10) depends on k in the following way: if $k > 0$, we employ the classical Sommerfeld condition; if $k = 0$, we seek solutions $u \in W(\Omega)$.

Theorem 1. *Problem 1 has at most one solution.*

Proof. For $k = 0$, we construct a variational formulation using domains $\{\Omega_j\}_{j=1}^M$, defined as in Assumption 2 and Green's formula for those domains. Then, we can employ the equivalence of norms for the homogeneous condition ($\mathbf{g} = 0$). For $k \geq 0$, the proof follows from analytic continuation (see [11] for more details). □

3 Boundary integral equation formulation

As it was previously mentioned, the unboundedness nature of Ω calls for boundary integral methods, converting the partial differential problem from the domain Ω to the boundary Γ. Let $G_k(\mathbf{x}, \mathbf{y})$ denote the free space fundamental solution, whose explicit representation is well known and can be found in [12, Section 3.1]. This symmetric satisfies

$$(-\Delta_{\mathbf{x}} - k^2) G_k(\mathbf{x}, \mathbf{y}) = \delta_{\mathbf{x}}(\mathbf{y}), \quad \forall\, k \geq 0, \tag{11}$$

where the derivatives operators are defined in distributional sense. For a fixed $\mathbf{x}$, if we restrict $\mathbf{y}$ to a set whose closure has positive distance to $\mathbf{x}$ then $G_k(\mathbf{x}, \mathbf{y})$ is an analytic function. For an unknown $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_M)$ defined in $\Gamma_1 \times \ldots \times \Gamma_M$, we search for a solution of Problem 1 constructed as

$$u(\mathbf{x}) = \int_\Gamma G_k(\mathbf{x}, \mathbf{y}) \boldsymbol{\lambda}(\mathbf{y}) d\mathbf{y} =: \sum_{i=1}^M (\mathsf{SL}_i[k]\lambda_i)(\mathbf{x}), \quad \forall\, \mathbf{x} \in \Omega, \tag{12}$$

wherein

$$(\mathsf{SL}_i[k]\lambda_i)(\mathbf{x}) := \int_{\Gamma_i} G_k(\mathbf{x}, \mathbf{y}) \lambda_i(\mathbf{y}) d\mathbf{y}, \tag{13}$$

denotes the single layer potential generated at a curve Γ_i. Though one can show that $(-\Delta_{\mathbf{x}} - k^2) u(\mathbf{x}) = 0$, for $\mathbf{x} \in \Omega$, we still need to show that there exists at least one $\boldsymbol{\lambda}$ such that $\gamma_j u = g_j$, and u has the correct behavior at infinity. To prove existence and uniqueness of $\boldsymbol{\lambda}$, we introduce some proprieties of boundary integral potentials.

Proposition 1. *For each arc Γ_i, with $i \in \{1, \ldots, M\}$, and $k \geq 0$, the single layer potential $\mathsf{SL}_i[k] : \widetilde{H}^{-\frac{1}{2}}(\Gamma_i) \to H^1_{loc}(\mathbb{R}^2)$ is a linear bounded map. For $k > 0$ and $\lambda_i \in \widetilde{H}^{-\frac{1}{2}}(\Gamma_i)$, the potential $u = \mathsf{SL}_i[k]\lambda_i$ fulfills the Sommerfeld radiation condition. For the Laplace case, it holds that $\mathsf{SL}_i[0] : \widetilde{H}^{-\frac{1}{2}}_{\langle 0 \rangle}(\Gamma_i) \to W(\mathbb{R}^2 \setminus \overline{\Gamma}_i)$.*

Proof. Boundedness is a direct consequence of the integration domain and the trivial equality

$$\mathsf{SL}_i[k]\lambda_i = \mathsf{SL}_{\partial\Omega_i}[k]\tilde{\lambda}_i, \tag{14}$$

where $\tilde{\lambda}_i$, is the extension by zero of λ_i over the boundary of the associated Ω_i. The mapping propriety for $\mathsf{SL}_i[0]$ can be obtained from the asymptotic behaviur (see [10, Corollary 8.11]),

$$(\mathsf{SL}_i[0]u)(\mathbf{x}) = -\frac{1}{2\pi} \langle u, 1 \rangle \log \|\mathbf{x}\|_2 + \mathcal{O}(\|\mathbf{x}\|_2^{-1}), \quad \|\mathbf{x}\|_2 \to \infty. \tag{15}$$

with the first term vanishing. □

As a direct consequence of the last result, we can define the boundary integral operator:

$$\mathcal{L}_{ij}[k] = \gamma_i \mathsf{SL}_j[k]. \tag{16}$$

Proposition 2 ([11])**.** *The following properties hold*

1. *For $k \geq 0$, the operator $\mathcal{L}_{ii}[k] : \widetilde{H}^{-\frac{1}{2}}(\Gamma_i) \to H^{\frac{1}{2}}(\Gamma_i)$ is linear and bounded for $i \in \{1, \ldots, M\}$.*

2. *For $k = 0$, let $\lambda_i \in \widetilde{H}^{-\frac{1}{2}}_{\langle 0 \rangle}(\Gamma_i)$. Then, there exist constants $c_{e,i} > 0$ such that*

$$\langle \mathcal{L}_{ii}[0]\lambda_i, \lambda_i \rangle_{\Gamma_i} \geq c_{e,i} \|\lambda_i\|^2_{\widetilde{H}^{-\frac{1}{2}}(\Gamma_i)}, \quad i = 1, \ldots, M. \tag{17}$$

3. *For $i \in \{1, \ldots, M\}$ and $k \geq 0$, there exist constants $c_{e,i} > 0$ and compact boundary operators $\mathcal{K}_{ii}[k] : \widetilde{H}^{-\frac{1}{2}}(\Gamma_i) \to H^{\frac{1}{2}}(\Gamma_i)$, such that*

$$\langle (\mathcal{L}_{ii}[k] + \mathcal{K}_{ii}[k])\lambda_i, u \rangle_{\Gamma_i} \geq c_{e,i} \|u\|^2_{\widetilde{H}^{-\frac{1}{2}}(\Gamma_i)}, \qquad \forall \lambda_i \in \widetilde{H}^{-\frac{1}{2}}(\Gamma_i). \tag{18}$$

4. *Assume that k is not an eigenvalue of the Laplace operator with Dirichlet conditions, for any domain enclosed by $\widetilde{\Gamma}_i$. Then, the self-interaction operators $\mathcal{L}_{ii}[k] : \widetilde{H}^{-\frac{1}{2}}(\Gamma_i) \to H^{\frac{1}{2}}(\Gamma_i)$ are coercive and injective for $k > 0$, and elliptic for $k = 0$ in $\widetilde{H}^{-\frac{1}{2}}_{\langle 0 \rangle}(\Gamma_i)$, for $i \in \{1, \ldots, M\}$.*

5. *For $k \geq 0$ the cross-interaction operators $\mathcal{L}_{ij}[k] : \widetilde{H}^{-\frac{1}{2}}(\Gamma_j) \to H^{\frac{1}{2}}(\Gamma_i)$ defined over disjoint interfaces are compact for $k \geq 0$, for all $(i, j) \in \{1, \ldots, M\} \times \{1, \ldots, M\}$ with $i \neq j$.*

For $M > 1$ arcs, we can formulate the problem as a boundary integral equation, to that end we denote by $\mathcal{L}[k]$ the operator matrix, with coefficients $(\mathcal{L}[k])_{ij} = \mathcal{L}_{ij}[k]$.

Problem 2 (Boundary Integral Problem)**.** *For $k > 0$, let $\mathbf{g} \in \mathbb{H}^{\frac{1}{2}}(\Gamma)$. We seek $\boldsymbol{\lambda} \in \widetilde{\mathbb{H}}^{-\frac{1}{2}}(\Gamma)$ such that*

$$\mathcal{L}[k]\boldsymbol{\lambda} = \mathbf{g}. \tag{19}$$

In the case $k = 0$, we look for $\boldsymbol{\lambda} \in \widetilde{\mathbb{H}}^{-\frac{1}{2}}_{\langle 0\rangle}(\Gamma)$.

The following properties concerning the operator $\mathcal{L}[k]$ are proved in [11].

Proposition 3. *Let $\mathcal{L}[k] : \widetilde{\mathbb{H}}^{-\frac{1}{2}}(\Gamma) \to \mathbb{H}^{\frac{1}{2}}(\Gamma)$, we have that for $k \geq 0$ is a bounded linear map. Moreover, for $k > 0$, $\mathcal{L}[k]$ is coercive injective whereas for $k = 0$, the map is coercive and injective in the subspace $\widetilde{\mathbb{H}}^{-\frac{1}{2}}_{\langle 0\rangle}(\Gamma)$.*

The proof uses Theorem 1 and so we need different spaces for the cases $k > 0$ and $k = 0$ in order to fulfill the condition at infinity. By the classical Fredholm alternative [12, Theorem 2.1.36], we can conclude the existence and uniqueness of Problem 2 which also proves the existence of Problem 1.

Theorem 2. *Problem 2 has a unique solution.*

In what follows we will denote

$$\mathcal{H}[k] := \begin{cases} \widetilde{\mathbb{H}}^{-\frac{1}{2}}_{\langle 0\rangle}(\Gamma) & \text{for } k = 0, \\ \widetilde{\mathbb{H}}^{-\frac{1}{2}}(\Gamma) & \text{for } k > 0. \end{cases} \tag{20}$$

4 Numerical Analysis

In this section, we describe a family of finite dimensional spaces to perform a Galerkin discretization of the boundary integral equation in Problem 2, and then study the approximation for the problem solution.

Let $\mathbb{P}_N(\Gamma_i)$ denote the space of polynomials of degree lower or equal to N, parametrized on Γ_i. Thus, for each $p \in \mathbb{P}_N(\Gamma_i)$, there is q a polynomial in $[-1, 1]$, such that $p = q \circ \mathbf{r}_i^{-1}$. We define the space:

$$\hat{\mathbb{P}}_N(\Gamma_i) := \left\{ p : p = \frac{q(\mathbf{x})}{|\mathbf{r}_i' \circ \mathbf{r}_i^{-1}(\mathbf{x})|} \;, \quad q \in \mathbb{P}_N(\Gamma_i) \right\}. \tag{21}$$

This space does not take into account the singularities at the end points of the arcs, so the following space offers a better alternative,

$$\hat{\mathbb{Q}}_N(\Gamma_i) := \left\{ w_i^{-1} p : p \in \hat{\mathbb{P}}_N(\Gamma_i) \right\}, \tag{22}$$

where the weight $w_i(\mathbf{x}) = \sqrt{1 - (\mathbf{r}_i^{-1}(\mathbf{x}))^2}$. We also denote by $\hat{\mathbb{Q}}_{N,\langle 0\rangle}(\Gamma_i)$ the subspace excluding constant polynomials. Notice that if we choice a basis for the in $[-1, 1]$, it defines a basis for $\hat{\mathbb{Q}}_N(\Gamma_i)$ in a natural way. In what follows, we will fix the basis for the polynomials as first kind Chebyshev polynomials, denoted by $\{T_n(t)\}_{n=0}^N$ [5, Section 4.1.2]. The corresponding basis in $\hat{\mathbb{Q}}_N(\Gamma_i)$ will be denoted $\{\phi_i^n\}_{n=0}^N$. Hence, the discrete problem can be written as

Problem 3. *For $k \geq 0$, given $N \in \mathbb{N}_0$, and $\mathbf{g} \in \mathbb{H}^{\frac{1}{2}}(\Gamma)$ we seek $\underline{\boldsymbol{\lambda}_N} \in \mathbb{C}^{M(N+1)}$ such that*

$$\mathsf{L}[k]\underline{\boldsymbol{\lambda}_N} = \mathbf{b}, \tag{23}$$

where $\mathsf{L}[k] \in \mathbb{C}^{M(N+1)\times M(N+1)}$ with entries $(L[k]_{ij})_{lm} = \left\langle \mathcal{L}_{ij}[k]\phi_i^m, \phi_j^l \right\rangle$, $\mathbf{b} \in \mathbb{C}^{M(N+1)}$ with components $b_i^l = \left\langle \mathbf{g}, \boldsymbol{\phi}_i^l \right\rangle$.

The approximation of Problem 2 is denoted by $\boldsymbol{\lambda}_N$ and is obtained as

$$(\lambda_N)_i = \sum_{n=0}^{N} (\underline{\lambda_N})_i^n \phi_i^n, \quad i = 1, \dots, M. \tag{24}$$

The total discretization space is defined as

$$\mathcal{H}_N[k] := \begin{cases} \prod_{i=1}^{M} \hat{\mathbb{Q}}_{N,\langle 0 \rangle}(\Gamma_i) & \text{for } k = 0, \\ \prod_{i=1}^{M} \hat{\mathbb{Q}}_N(\Gamma_i) & \text{for } k > 0. \end{cases} \tag{25}$$

We state some proprieties of the discretization spaces, the proofs can be found in [11].

Proposition 4. *$\{\mathcal{H}_N[k]\}_N$ is a increasing family of subsets of $\mathcal{H}[k]$, such that the union is dense. If $\boldsymbol{\lambda}$ is the solution of Problem 2, and $\boldsymbol{\lambda}_N \in \mathcal{H}_N[k]$ is the solution approximation found by solving (22), then there exists $N_0 \in \mathbb{N}$ and $C > 0$ such that for all $N > N_0$, it holds*

$$\|\boldsymbol{\lambda} - \boldsymbol{\lambda}_N\|_{\mathcal{H}[k]} \leq C \inf_{\mathbf{q}_N \in \mathcal{H}_N[k]} \|\boldsymbol{\lambda} - \mathbf{q}_N\|_{\mathcal{H}[k]}. \tag{26}$$

The following theorem shows how well we can approximate the solution of the boundary integral equations by an element of the discrete space $\mathcal{H}_N[k]$.

Theorem 3. *Let $m \in \mathbb{N}^*$, $N \in \mathbb{N}$. Assume Γ is the union of $\mathcal{C}^m$-arcs and $\mathbf{g} \in \mathcal{C}^m(\Gamma)$, denote $\boldsymbol{\lambda}$ the solution of Problem 2. Then, there exists $\boldsymbol{\lambda}_N \in \mathcal{H}_N[k]$ such that:*

- *If $m > 1$ and $N > m - 1$, it holds*

$$\|\boldsymbol{\lambda} - \boldsymbol{\lambda}_N\|_{\mathcal{H}[k]} \leq C_1 N^{-m+1}. \tag{27}$$

- *If $m = \infty$, then*

$$\|\boldsymbol{\lambda} - \boldsymbol{\lambda}_N\|_{\mathcal{H}[k]} \leq C_2 \sqrt{N} \rho^{-N}, \quad \rho > 1. \tag{28}$$

In the above, C_1 and C_2 are positive constants depending on Γ, and $\mathbf{g}$.

5 Linear system construction

We now show how all computations required to solve the problem 3 can be reduced to integrations over the canonical segment $\hat{\Gamma} = (-1, 1) \times \{0\}$. Once the integration domain is fixed, we show that the integration kernel can be split in a singular and regular parts, where for the singular one closed expressions are available.

By definition, it holds

$$(L[k]_{ij})_{lm} = \left\langle \mathcal{L}[k]\phi_i^m, \phi_j^l \right\rangle_{\Gamma_j} = \left\langle \mathcal{L}_{ij}[k]\phi_i^m, \phi_j^l \right\rangle_{\Gamma_j}. \tag{29}$$

Using the parametrizations $\mathbf{r}_i$ and $\mathbf{r}_j$ we have

$$(L[k]_{ij})_{lm} = \left\langle \hat{\mathcal{L}}_{ij}[k](\phi_i^m \circ \mathbf{r}_i)\|\mathbf{r}_i'\|_2, (\phi_j^l \circ \mathbf{r}_j)\|\mathbf{r}_j\|_2 \right\rangle_{\hat{\Gamma}}, \tag{30}$$

where $\hat{\mathcal{L}}_{ij}[k]$ corresponds to the operator parametrized over the respective arcs. By the definition of the discrete space, we have

$$(L[k]_{ij})_{lm} = \left\langle \hat{\mathcal{L}}_{ij}[k]w^{-1}T_m, w^{-1}T_l \right\rangle_{\hat{\Gamma}}, \tag{31}$$

where now $w(t) = \sqrt{(1-t^2)}$, and T_n denotes the n-th Chebyshev polynomial.

If $j \neq i$, the kernel function associated to this operator is analytic, and consequently, we can expand it as a Chebyshev series using the FFT and compute integrals using the same techniques as in [7]. If $i = j$, we employ the decomposition:

$$(L[k]_{ij})_{lm} = \left\langle \mathcal{L}^r[0]w^{-1}T_m, w^{-1}T_l \right\rangle_{\hat{\Gamma}} + \left\langle (\hat{\mathcal{L}}_{ij}[k] - \mathcal{L}^r[0])w^{-1}T_m, w^{-1}T_l \right\rangle_{\hat{\Gamma}}, \tag{32}$$

where $\mathcal{L}^r[0]$ has as kernel $G_0(x, y)$ with a parametrization associated with $\hat{\Gamma}$. Thus, the first term on the right-hand side can be computed combining the following expression

$$G_0(\mathbf{x}, \mathbf{y}) = -\frac{1}{2\pi}\log|t-s| = \frac{1}{2\pi}\log 2 + \sum_{n\geq 1}\frac{1}{n}T_n(t)T_n(s), \tag{33}$$

and the orthogonality relation $\langle w^{-1}T_m, T_l \rangle_{\hat{\Gamma}} = C_m \delta_{ml}$, where C_m is known. For the second term, the next result shows that it can be integrated as in the case $i \neq j$.

Proposition 5. *If Γ_j is at least $\mathcal{C}^2$, the operator $\hat{\mathcal{L}}_{jj}[k] - \mathcal{L}^R[0] : \widetilde{H}^{-\frac{1}{2}}(\hat{\Gamma}) \to H^{\frac{1}{2}}(\hat{\Gamma})$ has a kernel that is at least $\mathcal{C}^1$.*

Proof. For $k > 0$, $\Gamma_i = \Gamma_j = \hat{\Gamma}$ the kernel differences can be written as

$$C|t-s|^2 \log|t-s| + o(|t-s|^2), \tag{34}$$

from where one deduces that it is $\mathcal{C}^1$. If $k = 0$ and the arcs are not the same to $\hat{\Gamma}$, the result is given by computing the derivative of $\log|t-s| - \log\|\mathbf{r}_i(t) - \mathbf{r}_j(s)\|_2$. This two cases and expansions of the Helmholtz kernel yield the desired result. $\square$

6 Preconditioning

We use the result from the last section to construct a second kind formulation, whose discretization can be approximated by a preconditioned version of the linear system in Problem 3. A direct consequence of these results is that any preconditioner for the Laplace problem in $\hat{\Gamma}$ can be extended to work for problem 3.

Let us denote:

$$\mathcal{L}^R[0] := \begin{bmatrix} \mathcal{L}^r[0] & 0 & \dots & 0 \\ 0 & \mathcal{L}^r[0] & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathcal{L}^r[0] \end{bmatrix}, \quad \hat{\mathcal{L}}[k] := \begin{bmatrix} \hat{\mathcal{L}}_{11}[k] & \hat{\mathcal{L}}_{12}[k] & \dots & \hat{\mathcal{L}}_{1M}[k] \\ \hat{\mathcal{L}}_{21}[k] & \hat{\mathcal{L}}_{22}[k] & \dots & \hat{\mathcal{L}}_{2M}[k] \\ \vdots & \vdots & \ddots & \vdots \\ \hat{\mathcal{L}}_{M1}[k] & \hat{\mathcal{L}}_{M2}[k] & \dots & \hat{\mathcal{L}}_{MM}[k] \end{bmatrix}. \quad (35)$$

Notice that the discretization of $\hat{\mathcal{L}}[k]$ produces the same matrix of Problem 3. The following result is a direct consequence of Proposition 5.

Theorem 4. *If Γ is at least of class $\mathcal{C}^2$, then the operator*

$$\hat{\mathcal{L}}[k] - \mathcal{L}^R[0] : \widetilde{\mathbb{H}}^{-\frac{1}{2}}\left(\prod_{j=1}^{M}\hat{\Gamma}\right) \to \mathbb{H}^{\frac{1}{2}}\left(\prod_{j=1}^{M}\hat{\Gamma}\right) \quad (36)$$

is compact. The same result holds for $k = 0$ with domain $\widetilde{\mathbb{H}}^{-\frac{1}{2}}_{\langle 0\rangle}(\prod_{j=1}^{M}\hat{\Gamma})$.

Now, it is natural to consider the following operator decomposition:

$$\hat{\mathcal{L}}[k] = (\hat{\mathcal{L}}[k] - \mathcal{L}^R[0]) + \mathcal{L}^R[0], \quad (37)$$

which leads to the second kind formulation:

$$\left(\mathcal{L}^R[0]\right)^{-1}\hat{\mathcal{L}}[k]\hat{\boldsymbol{\lambda}} = \left[\mathcal{I} + \left(\mathcal{L}^R[0]\right)^{-1}(\hat{\mathcal{L}}[k] - \mathcal{L}^R[0])\right]\hat{\boldsymbol{\lambda}} = \left(\mathcal{L}^R[0]\right)^{-1}\hat{\mathbf{g}}, \quad (38)$$

where $\mathcal{I}$ denotes the identity operator. The discretization of $\left(\mathcal{L}^R[0]\right)^{-1}\hat{\mathcal{L}}[k]$ can be approximated by the preconditioned linear system:

$$\mathsf{L}^R[0]^{-1}\mathsf{L}[k]\underline{\boldsymbol{\lambda}_N} = \mathsf{L}^R[0]^{-1}\mathbf{b}, \quad (39)$$

where $\mathsf{L}[k]$ and $\mathbf{b}$ are defined as in Problem 3, and $\mathsf{L}^R[0]^{-1}$ is the inverse of the discretization of $\mathcal{L}^R[0]$. Notice that using the Chebyshev basis, the matrix $\mathsf{L}^R[0]$ is diagonal and can be easily inverted.

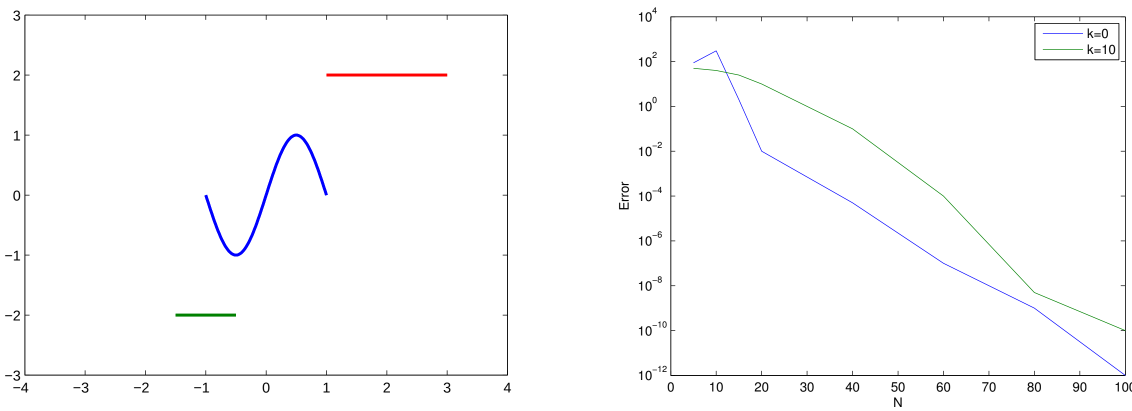
7 Matrix Compresion

We have shown that the coefficients of the matrix in Problem 3 are of the form

$$(L[k]_{ij})_{lm} = \left\langle \hat{\mathcal{L}}_{ij}[k]w^{-1}T_m, w^{-1}T_l \right\rangle_{\hat{\Gamma}}. \quad (40)$$

We recognize that this correspond to the l-th term in the Chebyshev expansion of the function $\hat{\mathcal{L}}_{ij}[k]w^{-1}T_m$. It is well known that there exists a connection between the

Figure 1: Test Case



(a) Geometrical arrangement of arcs

(b) Error converge in $\widetilde{H}^{-\frac{1}{2}}$-norm for Laplace ($k = 0$) and Helmholtz ($k = 10$) cases.

rate of decay of the coefficients in Fourier-Chebyshev expansion and the smoothness of the function [16, Chapter 7 and 8]. In particular, when the function is analytic the coefficients exhibit exponential decay. Hence, we know that for $i \neq j$ the kernel function is analytic and so it is the function $\hat{\mathcal{L}}_{ij}[k]w^{-1}T_m$. Hence, we can compute only a few terms and discard the rest knowing that they will decay exponentially. A practical algorithm that use this idea is presented in [11].

8 Numerical Results

For our numerical experiments, we first show convergence results for a configuration of three different arcs as is shown in Figure 1a. We compute the $\mathcal{H}[k]$ norm of the absolute value for $k = 10$, with overkill solution taken by approximating with $N = 120$ polynomials per segment. Results are showed in Figure 1b confirming our findings.

Next, we show the performances of the proposed preconditioner and matrix compression techniques for different wavenumbers. The results are summarized in Table 1: *Size* refers to the length of the columns in the matrix $\mathsf{L}[k]$; *%Nnz* is the percentage of entries of the matrix that are nonzero after compression is applied, *RelError* denotes the relative error when comparing the solutions of the full linear system and the compressed one. The next three columns in Table 1 are related to the number of GMRES iterations needed to obtain a residual bounded by $1e-8$. The first one, *No Pre.*, indicates no preconditioner is applied, $L^R[0]$ is the preconditioner detailed in this work; and, finally *Diag.* is the diagonal preconditioner.

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Table 1: Preconditioner and Compresion results

		$k = 0$			
Size	%Nnz	RelError	No Pre.	$L^R[0]$	Diag.
60	51.7	2.45e-11	27	14	13
240	15.1	7.49e-9	31	14	13
480	5.92	7.57e-9	31	14	13
		$k = 5$			
Size	%Nnz	RelError	No Pre.	$L^R[0]$	Diag.
93	54.9	1.29e-10	41	31	28
273	19.1	2.71e-8	42	31	28
513	7.99	2.72e-8	42	31	28
		$k = 20$			
Size	%Nnz	RelError	No Pre.	$L^R[0]$	Diag.
363	28.6	4.84e-8	93	79	61
603	15.8	1.16e-5	93	79	61
1083	19.8	7.85e-8	93	79	61
		$k = 100$			
Size	%Nnz	RelError	No Pre.	$L^R[0]$	Diag.
963	29.4	5.84e-8	246	156	157
1803	16.1	1.10e-7	251	160	159
2523	11.0	1.10e-7	251	160	159

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