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L-SCHEME AND NEWTON BASED SOLVERS FOR A NONLINEAR BIOT MODEL

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Abstract. In this work we propose convergent iterative solvers based on the L-scheme and Newton's method for a nonlinear Biot model. More precisely, in the mechanics equation one Lamé coefficient and in the fluid equation the compressibility are assumed to be nonlinear. The nonlinearities are satisfying certain assumptions, in particular they are supposed to be monotonically increasing. We use the Newton method and the L-scheme for linearizing the equations. Monolithical and fixed-stress type splitting approaches are proposed. The solvers can be applied to any spatial discretization. The theoretical convergence of the schemes is discussed.

1 INTRODUCTION

The term poromechanics is commonly used to denote the fully coupled system consisting of porous media flow and mechanics. It has plenty of currently important applications, e.g. geothermal energy, CO_2 storage, biotechnology or pharmaceutics, to name few. The most used mathematical model for poromechanics is the (quasi-static) linear Biot model [17]. In this work we consider a nonlinear extension of the linear Biot model, as presented in [8]:

$$-\nabla \cdot [2\mu\varepsilon(\mathbf{u}) + h(\nabla \cdot \mathbf{u})] + \alpha\nabla \cdot (pI) = \mathbf{f}, \tag{1}$$

$$\partial_t \left(b(p) + \alpha\nabla \cdot \mathbf{u}\right) - \nabla \cdot (k\nabla p) = g, \tag{2}$$

where $\mathbf{u}$ is the displacement, $\mu > 0$ is the constant shear modulus, $\varepsilon(\mathbf{u}) = \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^t\right)$ is the linearized strain, p is the fluid pressure, $\mathbf{I}$ is the identity matrix, α the Biot coefficient and k is the hydraulic conductivity. For simplicity, the gravitational effects are neglected. The coefficient functions $h(\cdot)$ and $b(\cdot)$, as well as the source terms $\mathbf{f}, g$ are supposed to be given. Initial and boundary conditions are completing the model. The linear Biot model is a particular case of (1)-(2), obtained by choosing the function $h(\cdot)$ and $b(\cdot)$ to be the identity. A very important assumption is that $h(\cdot)$ and $b(\cdot)$ are monotonically increasing and Lipschitz continuous. For a justification of the validity of the considered model we refer to the book [38]. We remark that the results of the present paper can be easily extended to a model containing a nonlinearity of the form $a(\nabla \mathbf{u})$ in the mechanics equation, as long as it is monotone increasing and Lipschitz continuous and the coupling term remains linear.

Independent of the spatial and implicit temporal discretizations one chooses, solving the system (1)-(2) is very challenging. This is due to the coupling and the nonlinearities. In the present work we present monolithical and splitting approaches for solving this nonlinear system. As splitting approach, we will use the well-recognized fixed-stress scheme, see [36, 22]. For the convergence analysis of the fixed-stress method applied to the linear Biot model we refer to [26, 11, 6, 7]. For a discussion on the stabilization/tuning parameter used in the fixed-stress approach we refer to [15, 11]. A theoretical investigation on the optimal choice for this parameter is performed in [37]. The linearization is based on either Newton method, or the L-scheme [25, 30, 32, 33] or on a combination of them [25, 12]. For monolithic and splitting schemes based solely on L-scheme for the system (1)-(2) we refer to [8].

These techniques can be applied as well to multiphase flow and mechanics, see [15, 12, 23, 21]. In [24] the fixed-stress method is applied in connection with fracture propagation and phase field models [24]. The fixed-stress method can be also used as a smoother for multigrid approaches [20] or as pre-conditioner [8, 39]. For multirate and multiscale fixed-stress schemes we refer to [4] and [18], respectively. Recently, a parallel-in-time fixed-stress scheme for the linear Biot model was proposed in [9].

2 SOLVERS FOR THE NONLINEAR BIOT MODEL

In this section we present some monolithic and splitting schemes for solving the nonlinear system (1)-(2). We begin by discussing the spatial and temporal discretizations, then we give the linearization ideas and finally present the schemes.

There is a huge literature on discretization methods for the linear Biot model. Backward Euler is the most common discretization in time, see e.g. [27, 11]. Higher-order discretizations in time are presented and analysed in [6] (see also [5] for details on space-time techniques). Many different combinations of spatial discretizations for flow and mechanics were proposed in the last years. We mention continuous Galerkin for the mechanics and mixed finite elements for the flow [29, 42, 11, 3], the MINI element [34], non-conforming finite elements [40], mixed finite elements for flow and mechanics [42, 1], cell-centered finite volumes [28], or discontinuous Galerkin [16]. Higher-order space-time elements were considered in [6, 7]. We refer to the recent paper [35] for a discussion on the stability of discretizations for Biot's model. We finally mention that adaptive techniques were proposed e.g. in [19, 1].

A key ingredient in designing monolithic or splitting schemes for the nonlinear system (1)-(2) is the linearization method. We propose as linearization either the L-scheme, e.g. [25] or the Newton method. The idea of the L-scheme, see [25, 30] is to solve a nonlinearity

$F(U)$ iteratively by linearizing in the following way:

$$F(U^i) + L(U^{i+1} - U^i), \tag{3}$$

where i is the iteration index and $L > 0$ a free to be chosen stabilization (or tuning) parameter. When $i \to \infty$ we must have $U^i \to U$, ensuring obviously the consistency of the scheme. The L-scheme can be interpreted as either a stabilized Picard method or as quasi-Newton method. The L-scheme is very robust but only linearly convergent. It can be applied to non-smooth, but monotonically increasing functions $F(\cdot)$. For the case of Hölder continuous (not Lipschitz) functions $F(\cdot)$ we refer to [13, 33]. The L-scheme can be speeded up by using the Anderson acceleration [2, 15]. The main advantages of the L-scheme are

- It does not involve the computations of derivatives.
- The arising linear systems are well-conditioned.
- It can be applied to non-smooth nonlinearities.
- It is very easy to be understood and implemented.

The second linearization method proposed in this paper is the well-recognized Newton method. In this case, a nonlinearity $F(U)$ will be solved by

$$F(U^i) + F'(U^i)(U^{i+1} - U^i), \tag{4}$$

with F' being the Jacobian matrix. The Newton method is quadratic convergent, but the convergence is local. This means that the starting value for the iterations should not be too far from the (unknown) solution. To increase the robustness of the Newton method, one can perform first some L-scheme iterations and then switch to Newton [25]. Another way to increase the robustness of Newton's method is by applying the Anderson acceleration [15].

We can now present some iterative solvers for the nonlinear system (1)-(2). Backward Euler is applied for the temporal discretization. A three-field formulation is considered for the spatial discretization, but the solvers can be applied to any discretization. In the following n denotes always the time level, while i stays for the iteration index.

A monolithic Newton solver

$$\begin{aligned}
-\nabla \cdot [2\mu\varepsilon(\mathbf{u}^{n,i+1}) + h(\nabla \cdot \mathbf{u}^{n,i}) + h'(\nabla \cdot \mathbf{u}^{n,i})(\mathbf{u}^{n,i+1} - \mathbf{u}^{n,i}) - \alpha p^{n,i+1} I] &= \mathbf{f}, \\
b(p^{n,i}) + b'(p^{n,i})(p^{n,i+1} - p^{n,i}) + \alpha \nabla \cdot (\mathbf{u}^{n,i+1} - \mathbf{u}^{n-1}) + \tau \nabla \cdot \mathbf{q}^{n,i+1} &= \tau S_f + b(p^{n-1}), \\
\mathbf{q}^{n,i+1} + k \nabla p^{n,i+1} &= g.
\end{aligned}$$

The following scheme was presented and analyzed in [8]:

A splitting L-scheme
Step 1:

$$\begin{aligned} b(p^{n,i}) + L_1(p^{n,i+1} - p^{n,i}) + \alpha\nabla\cdot(\mathbf{u}^{n,i} - \mathbf{u}^{n-1}) + \tau\nabla\cdot\mathbf{q}^{n,i+1} &= \tau S_f + b(p^{n-1}), \\ \mathbf{q}^{n,i+1} + k\nabla p^{n,i+1} &= g. \end{aligned}$$

Step 2:

$$-\nabla\cdot[2\mu\varepsilon(\mathbf{u}^{n,i+1}) + h(\nabla\cdot\mathbf{u}^{n,i}) + L_2(\mathbf{u}^{n,i+1} - \mathbf{u}^{n,i}) - \alpha p^{n,i+1}I] = \mathbf{f},$$

where the stabilization parameters $L_1, L_2 \geq 0$ are free to be chosen.

From the above scheme, by introducing the Jacobian matrices one obtains a splitting Newton scheme:

A splitting Newton solver
Step 1:

$$\begin{aligned} b(p^{n,i}) + b'(p^{n,i})(p^{n,i+1} - p^{n,i}) + \alpha\nabla\cdot(\mathbf{u}^{n,i} - \mathbf{u}^{n-1}) + \tau\nabla\cdot\mathbf{q}^{n,i+1} &= \tau S_f + b(p^{n-1}), \\ \mathbf{q}^{n,i+1} + k\nabla p^{n,i+1} &= g. \end{aligned}$$

Step 2:

$$-\nabla\cdot[2\mu\varepsilon(\mathbf{u}^{n,i+1}) + h(\nabla\cdot\mathbf{u}^{n,i}) + (L + h'(\nabla\cdot\mathbf{u}^{n,i}))(\mathbf{u}^{n,i+1} - \mathbf{u}^{n,i}) - \alpha p^{n,i+1}I] = \mathbf{f}.$$

where the stabilization parameter $L \geq 0$ is free to be chosen.

We refer to [8] for a monolithic L-scheme for nonlinear Biot. Similar splitting schemes, in which the nonlinear subproblems are solved in each step until convergence (i.e. not performing just one iteration as above), can be set up and analyzed [10]. The main convergence result can be summarized in the following theorem. For the proof, techniques from [11], [25] and [31] are combined. The convergence of the L-scheme based solvers has been proved in [8].

Theorem 1. *Assuming that the nonlinearities are Lipschitz continuous and monotone increasing, and that the stabilization parameters are big enough, then all the considered schemes are at least linearly convergent. The monolithic Newton scheme is quadratic convergent.*

Remark 1. *The schemes can be accelerated by combining the $L-$scheme with the Newton method as done in [25, 12] or by using the Anderson acceleration [15]. We point out that the Anderson acceleration has also a stabilization effect, as proved in [15].*

3 CONCLUSIONS

We considered a quasi-static, nonlinear Biot model. Different nonlinear solvers based on the L-scheme, Newton method and the fixed-stress splitting method were presented. The only quadratic convergent scheme is the monolithic Newton. The splitting Newton method requires also a stabilization parameter, otherwise the (linear) convergence can not be guaranteed. The analysis of the schemes and illustrative numerical experiments will be presented in [10].

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