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NUMERICAL MODELING OF HEAT TRANSFER IN BIOLOGICAL TISSUE DOMAIN USING THE FUZZY FINITE DIFFERENCE METHOD

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Abstract. In this paper, the numerical analysis of a heat transfer process proceeding in the non-homogeneous biological tissue domain is presented. A one-dimensional problem is considered. Additionally, in the mathematical model, the thermophysical parameters of each skin's layer (epidermis, dermis, subcutaneous region) such as volumetric specific heat, thermal conductivity, perfusion coefficient and metabolic heat source are given as fuzzy numbers. The base of mathematical model is given by a set of Pennes fuzzy equations supplemented by the adequate boundary and initial conditions. The problem discussed has been solved using the fuzzy finite difference method with α-cuts. The examples of numerical computations are presented in the final part of the paper.

1 INTRODUCTION

Thermophysical parameters of biological tissue (thermal conductivity, volumetric specific heat, perfusion coefficient etc.) can change in a wide range and this is caused by individual traits as age, sex, occupation etc. This fact suggests the application of fuzzy values of these parameters to mathematical model of process considered. Such an approach in which the parameters appearing in the mathematical model are treated as constant values is widely used [1,2,3]. Here, the fuzzy values of thermal conductivity, volumetric specific heat, perfusion coefficient and metabolic heat source are taken into account.

In the paper the bio-heat transfer proceeding in a one-dimensional skin tissue described by the governing equations is considered. The problem has been solved using a fuzzy version of the finite difference method (FDM) with α-cuts and the rules of directed interval arithmetic [4,5]. The application of α-cuts allows one to avoid very complicated arithmetical operations in the fuzzy numbers set because the α-cuts are closed intervals. The main advantage of the directed interval arithmetic upon the usual interval arithmetic is that the obtained temperature intervals are much narrower and their width does not increase in time [6,7].

In the final part of the paper the examples of numerical simulations are presented for the skin tissue domain subjected to external heat source.

2 THE FUZZY NUMBERS

The ground of the mathematical rules used in this paper is given by the fuzzy set theory. This approach is not common in solving heat transfer problems and that is why some of the definitions used in this concept must be explained [8].

First of all, the definition of a fuzzy set will be introduced. The fuzzy set $\tilde{A}$ in a non-empty universal set $\mathbb{X}$ ($\tilde{A} \subseteq \mathbb{X}$) can be expressed by a set of pairs consisting of the elements $x \in \mathbb{X}$ and a certain degree of pre-assumed membership $\mu_{\tilde{A}}(x)$ of the form

$$\tilde{A} = \left\{ \left(x, \mu_{\tilde{A}}(x) \right); \; x \in \mathbb{X} \right\} \tag{1}$$

where function $\mu_{\tilde{A}}(x)$ is defined as follows

$$\mu_{\tilde{A}} : \mathbb{X} \to [0, 1] \tag{2}$$

In fuzzy sets, each element is mapped to [0,1] by membership function $\mu_{\tilde{A}}(x)$, where [0,1] means real numbers between 0 and 1 (including 0 and 1). Consequently, a fuzzy set is a 'vague boundary set' compared with a crisp set.

For every $x \in \mathbb{X}$ can be considered three types of membership to the fuzzy set $\tilde{A}$:

1. $\mu_{\tilde{A}}(x) = 1$ – full membership to the fuzzy set, $x \in \tilde{A}$,

2. $\mu_{\tilde{A}}(x) = 0$ – lack of membership to the fuzzy set, $x \notin \tilde{A}$,

3. $0 < \mu_{\tilde{A}}(x) < 1$ – partial membership to the fuzzy set.

The α-cut set $\tilde{A}$ in universal set $\mathbb{X}$ is made up of members whose membership is not less than α for every $\alpha \in [0, 1]$ [8,9]

$$\tilde{A}_\alpha = \left\{ x \in \mathbb{X} \colon \mu_{\tilde{A}}(x) \geq \alpha \right\} \tag{3}$$

The value α is arbitrary and this α-cut set is a crisp set. This set is determined by the following characteristic function:

$$\chi_{\tilde{A}_\alpha} = \begin{cases} 1 & \text{for } \mu_{\tilde{A}}(x) \geq \alpha \\ 0 & \text{for } \mu_{\tilde{A}}(x) < \alpha \end{cases} \tag{4}$$

Every fuzzy set $\tilde{A}$ can be defined as a sum of all its α-cuts

$$\tilde{A} = \sum_{\alpha \in [0, 1]} \alpha \cdot \tilde{A}_\alpha \tag{5}$$

where $\alpha \cdot \tilde{A}_\alpha$ is a fuzzy set in the universe $\mathbb{X}$, whose membership function is the following

$$\mu_{\alpha \cdot \tilde{A}_\alpha} = \begin{cases} \alpha & \text{for } x \in \tilde{A}_\alpha \\ 0 & \text{for } x \notin \tilde{A}_\alpha \end{cases} \tag{6}$$

Arithmetical operations are generally very complicated. Among the infinite quantity of possible fuzzy sets that can be qualified as fuzzy numbers, some types of membership functions are of particular importance.

Due to its rather simple membership function of a linear type, triangular fuzzy numbers are one of the most frequently used fuzzy numbers and in this paper, triangular fuzzy numbers will be used to solve the problem analysed.

A triangular fuzzy number $\tilde{a}$ is a set with the following membership function [10]

$$\mu_{\tilde{a}}(x) = \begin{cases} 0, & x < a^- \\ \dfrac{x - a^-}{a_0 - a^-}, & a^- \le x \le a_0 \\ \dfrac{a^+ - x}{a^+ - a_0}, & a_0 \le x \le a^+ \\ 0, & x > a^+ \end{cases} \tag{7}$$

where a_0 is the core of the number, a^-, a^+ are the left and the right end of the number respectively. A triangular fuzzy number can be written as $\tilde{a} = \left(a^-, a_0, a^+\right)$.

One of the ways to avoid very complicated arithmetic operations performed on fuzzy numbers is to apply α-cuts of fuzzy numbers. In this case, the mathematical operations are defined according to the rules of the directed interval arithmetic performed for every α-cut.

The α-cut of a fuzzy number $\tilde{a}$ defined by a pair of functions $a^- : [0, 1] \to \mathbb{R}$ and $a^+ : [0, 1] \to \mathbb{R}$ is called a set of closed intervals [8]

$$\forall \alpha \in [0, 1] \qquad \tilde{a}_\alpha = \left[a_\alpha^-, a_\alpha^+\right] \tag{8}$$

which satisfies the following conditions

$$\begin{aligned} &1.\ a^- : \alpha \to a_\alpha^- \in \mathbb{R} \\ &2.\ a^+ : \alpha \to a_\alpha^+ \in \mathbb{R} \\ &3.\ a_\alpha^- \le a_\alpha^+ \end{aligned} \tag{9}$$

where a^- (a^+) is a limited, monotonic function for every $\alpha \in [0, 1]$.

Every fuzzy number can be presented as a sum of all its own α-cuts

$$\tilde{a} = \sum_{\alpha \in [0, 1]} \tilde{a}_\alpha \tag{10}$$

The α-cut of a triangular fuzzy number $\tilde{a} = \left(a^-,\, a_0,\, a^+\right)$ is the set of all closed intervals in the form

$$\forall\alpha \in [0,\, 1] \qquad \tilde{a}_\alpha = \left[\left(a_0 - a^-\right)\alpha + a^-,\, \left(a_0 - a^+\right)\alpha + a^+\right] \tag{11}$$

The decomposition of a fuzzy number allows one to make the mathematical operations on closed intervals which are α-cuts. In this situation, complicated arithmetic operations can be omitted and it is possible to apply the interval arithmetic for every α-cut. The mathematical operations are simplified, because they are done only on the ends of the intervals.

For the α-cuts of two fuzzy numbers $\tilde{a}$ and $\tilde{b}$, the following mathematical operations can be defined ($\forall\alpha \in [0,\, 1]$) [10]:

– addition

$$\left(\tilde{a} + \tilde{b}\right)_\alpha = \left[a_\alpha^- + b_\alpha^-,\, a_\alpha^+ + b_\alpha^+\right] \tag{12}$$

– subtraction

$$\left(\tilde{a} - \tilde{b}\right)_\alpha = \left[a_\alpha^- - b_\alpha^+,\, a_\alpha^+ - b_\alpha^-\right] \tag{13}$$

– multiplication

$$\left(\tilde{a} \cdot \tilde{b}\right)_\alpha = \left[\min\left\{a_\alpha^- b_\alpha^-,\, a_\alpha^- b_\alpha^+,\, a_\alpha^+ b_\alpha^-,\, a_\alpha^+ b_\alpha^+\right\},\, \max\left\{a_\alpha^- b_\alpha^-,\, a_\alpha^- b_\alpha^+,\, a_\alpha^+ b_\alpha^-,\, a_\alpha^+ b_\alpha^+\right\}\right] \tag{14}$$

– division ($0 \notin \left[b_\alpha^-,\, b_\alpha^+\right]$)

$$\left(\frac{\tilde{a}}{\tilde{b}}\right)_\alpha = \left[\min\left\{\frac{a_\alpha^-}{b_\alpha^-},\, \frac{a_\alpha^-}{b_\alpha^+},\, \frac{a_\alpha^+}{b_\alpha^-},\, \frac{a_\alpha^+}{b_\alpha^+}\right\},\, \max\left\{\frac{a_\alpha^-}{b_\alpha^-},\, \frac{a_\alpha^-}{b_\alpha^+},\, \frac{a_\alpha^+}{b_\alpha^-},\, \frac{a_\alpha^+}{b_\alpha^+}\right\}\right]_\alpha \tag{15}$$

Applying α-cuts of the fuzzy numbers allows one to use directed interval arithmetic.

3 FUZZY GOVERNING EQUATIONS

Bio-heat transfer proceeding in the heterogeneous skin tissue domain of the thickness L_3 can be described by the system of fuzzy energy equations written in the following form [5,11,12,13]

$$L_{e-1} < x < L_e: \ \tilde{c}_e \frac{\partial T_e(x,\ t)}{\partial t} = \tilde{\lambda}_e \frac{\partial^2 T_e(x,\ t)}{\partial x^2} + \tilde{Q}_e(x,\ t) \tag{16}$$

where $e = 1,\ 2,\ 3$ corresponds to the successive layers of skin such as epidermis, dermis, hypodermis (see Figure 1), $\tilde{c}_e$ is the fuzzy volumetric specific heat, $\tilde{\lambda}_e$ is the fuzzy thermal conductivity, $\tilde{Q}_e$ is the capacity of fuzzy internal heat sources, T_e is the temperature, x and t denote spatial co-ordinate and time.

The capacity of fuzzy internal heat sources is a sum of two components

$$\tilde{Q}_e(x,\ t) = \tilde{G}_{Be}\, c_B \left[T_B - T_e\ (x,\, t) \right] + \tilde{Q}_{me} \tag{17}$$

where $\tilde{G}_{Be}$ is the fuzzy perfusion coefficient, c_B is the volumetric specific heat of blood, T_B is the arterial blood temperature, $\tilde{Q}_{me}$ is the fuzzy metabolic heat source [14,15].

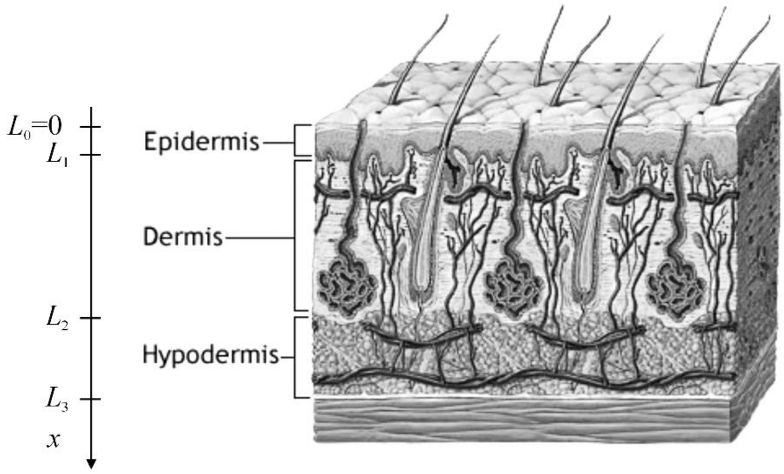

Figure 1: Skin tissue [16]

The fuzzy equations (16) must be supplemented by the boundary conditions

$$\begin{cases} x = 0: \quad \tilde{q}(x,\, t) = -\tilde{\lambda}_1 \dfrac{\partial T_1}{\partial n} = \tilde{q}_b \\[2ex] x = L_3: \quad \tilde{q}(x,\, t) = -\tilde{\lambda}_3 \dfrac{\partial T_3}{\partial n} = 0 \end{cases} \tag{18}$$

and the initial condition of the form

$$t = 0: \qquad T_e(x,\, 0) = T_0 \tag{19}$$

where $\tilde{q}_b$ is the given fuzzy value of the external heat source, T_0 is the initial temperature. Between the successive sub-domains the continuity condition is taken into account [5,11,12]

$$x = L_e: \quad \begin{cases} -\tilde{\lambda}_e \dfrac{\partial T_e(x,\, t)}{\partial n} = -\tilde{\lambda}_{e+1} \dfrac{\partial T_{e+1}(x,\, t)}{\partial n} \\[2ex] T_e(x,\, t) = T_{e+1}(x,\, t) \end{cases} \tag{20}$$

where e = 1, 2.

The equations (16) – (20) create the mathematical model of the process discussed.

The problem formulated has been solved by means of fuzzy finite difference method using α-cuts and the rules of directed interval arithmetic [4,5,10,17].

4 FUZZY FINITE DIFFERENCE METHOD

The problem analysed has been solved using the fuzzy FDM with α-cuts and the rules of directed interval arithmetic. At first, the time grid with a constant step $\Delta t = t^f - t^{f-1}$ is introduced

$$t^0 < t^1 < \ldots < t^{f-2} < t^{f-1} < t^f < \ldots < t^F < \infty \tag{21}$$

The left-hand side of the fuzzy energy equations (16) for the time t^f can be substituted by a differential quotient

$$\left(\tilde{c}_e \frac{\partial \tilde{T}_e(x,\ t)}{\partial t} \right)_i^{f-1} = \left(\tilde{c}_e \right)_i^{f-1} \frac{\left(\tilde{T}_e \right)_i^f - \left(\tilde{T}_e \right)_i^{f-1}}{\Delta t} \tag{22}$$

Additionally, the first term of the right-hand side of the energy equations can be transformed using the differential approximation of the second derivative

$$\left(\tilde{\lambda}_e \frac{\partial^2 \tilde{T}_e(x,\ t)}{\partial x^2} \right)_i^{f-1} = \left(\tilde{\lambda}_e \right)_i^{f-1} \frac{\left(\tilde{T}_e \right)_{i+1}^{f-1} - 2\left(\tilde{T} \right)_i^{f-1} + \left(\tilde{T}_e \right)_{i-1}^{f-1}}{\left(\Delta x_e \right)^2} \tag{23}$$

where Δx_e is the mesh step and i is the index of the central point of star [5].

Finally, one obtains the following differential interval equations

$$\begin{aligned} \left(\tilde{T}_e \right)_i^f = & \left(1 - 2\tilde{b}_e \right) \left(\tilde{T}_e \right)_i^{f-1} + \tilde{b}_e \left[\left(\tilde{T}_e \right)_{i+1}^{f-1} - \left(\tilde{T}_e \right)_{i-1}^{f-1} \right] \\ & + \frac{\Delta t}{\left(\tilde{c}_e \right)_i^{f-1}} \left\{ \left(\tilde{G}_{Be} \right)_i^{f-1} c_B \left[T_B - \left(\tilde{T}_e \right)_i^{f-1} \right] + \left(\tilde{Q}_{me} \right)_i^{f-1} \right\} \end{aligned} \tag{24}$$

where

$$\tilde{b}_e = \frac{\left(\tilde{\lambda}_e \right)_i^{f-1} \Delta t}{\left(\tilde{c}_e \right)_i^{f-1} \left(\Delta x_e \right)^2} \tag{25}$$

Using the equation (24) the temperature at the point i for time level f can be found under the assumption that the stability condition for explicit differential scheme is fulfilled [11].

The FDM approximation of the boundary conditions is constructed in a similar way as in the paper [18]. The 'boundary' nodes are located at the distance $0.5\Delta x_e$ with respect to the real boundary. This approach gives a better approximation of the Neumann and Robin boundary conditions.

The boundary condition of the fourth type is the condition of heat flow continuity at the contact of two subdomains ('e' and 'e+1'). The domain considered is covered by a regular geometrical mesh (Figure 2).

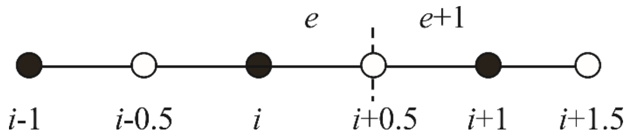

Figure 2: The regular geometrical mesh

The boundary condition of the fourth type for ideal contact can be written using the following equations

$$x = L_e : \quad \begin{cases} -\left(\tilde{\lambda}_e \dfrac{\partial \tilde{T}_e}{\partial n}\right)_{i+0.5}^{f-1} = -\left(\tilde{\lambda}_{e+1} \dfrac{\partial \tilde{T}_{e+1}}{\partial n}\right)_{i+0.5}^{f-1} \\ \left(\tilde{T}_e\right)_{i+0.5}^{f-1} = \left(\tilde{T}_{e+1}\right)_{i+0.5}^{f-1} \end{cases} \tag{26}$$

As mentioned, the mathematical manipulations leading to the designation of temperature field corresponding to time level f should be done using α-cuts according to the rules of directed interval arithmetic [4,6,19].

5 RESULTS OF COMPUTATIONS

As a numerical example the bio-heat transfer in a skin tissue of thickness L_3 = 12.1 mm has been analysed. The following input data have been introduced [11]: L_1 = 0.1 mm, L_2 = 2.1 mm, c_B = 3.9962·10^6 J/(m^3·K), T_B = 37°C, G_{B1} = 0, $G_{B2} = G_{B3}$ = 0.00125 (m^{3}blood/s)/m^{3}tissue, initial temperatures T_{10} = T_{20} = T_{30} = 37 °C, the external heat source q_b = 15·10^3 W/m^2 , the time step $\Delta t = 0.001$ s, the mesh step $\Delta x_e = (L_e - L_{e-1})/n_e$ where $n_1 = 5$, $n_2 = 30$ and $n_3 = 60$. The fuzzy triangular numbers of thermal conductivities $\tilde{\lambda}_e = (\lambda_e - 0.05\lambda_e, \lambda_e, \lambda_e + 0.05\lambda_e)$ volumetric specific heats $\tilde{c}_e = (c_e - 0.05c_e, c_e, c_e + 0.05c_e)$ and metabolic heat sources $\tilde{Q}_{me} = (Q_{me} - 0.05Q_{me}, Q_{me}, Q_{me} + 0.05\lambda_e)$ have been introduced (for e = 1, 2, 3) where λ_1 = 0.235 W/(m·K), λ_2 = 0.445 W/(m·K), λ_3 = 0.185 W/(m·K), c_1 = 4.3068·10^6 J/(m^3·K), c_2=3.96·10^6 J/(m^3·K), c_3 = 2.674·10^6 J/(m^3·K), $Q_{m1} = 0$ W/m^3 and $Q_{m2} = Q_{m3} = 245$ W/m^3. The time of external heat source exposition has been assumed as 5 s.

Figure 3 presents the heating and cooling curves at the selected internal nodes L_1 (1), L_2 (2) for chosen values of parameter α. It should be pointed out that for each node of the domain considered there are two curves representing the beginning and end of temperature intervals.

Figure 4 presents the interval values of temperatures for the chosen parameter α at the node corresponding to node L_1 after 20 s. This figure shows the decomposition of the triangular fuzzy number being the temperature value and the dependence between the value of the parameter α and the width of the resulting temperature interval.

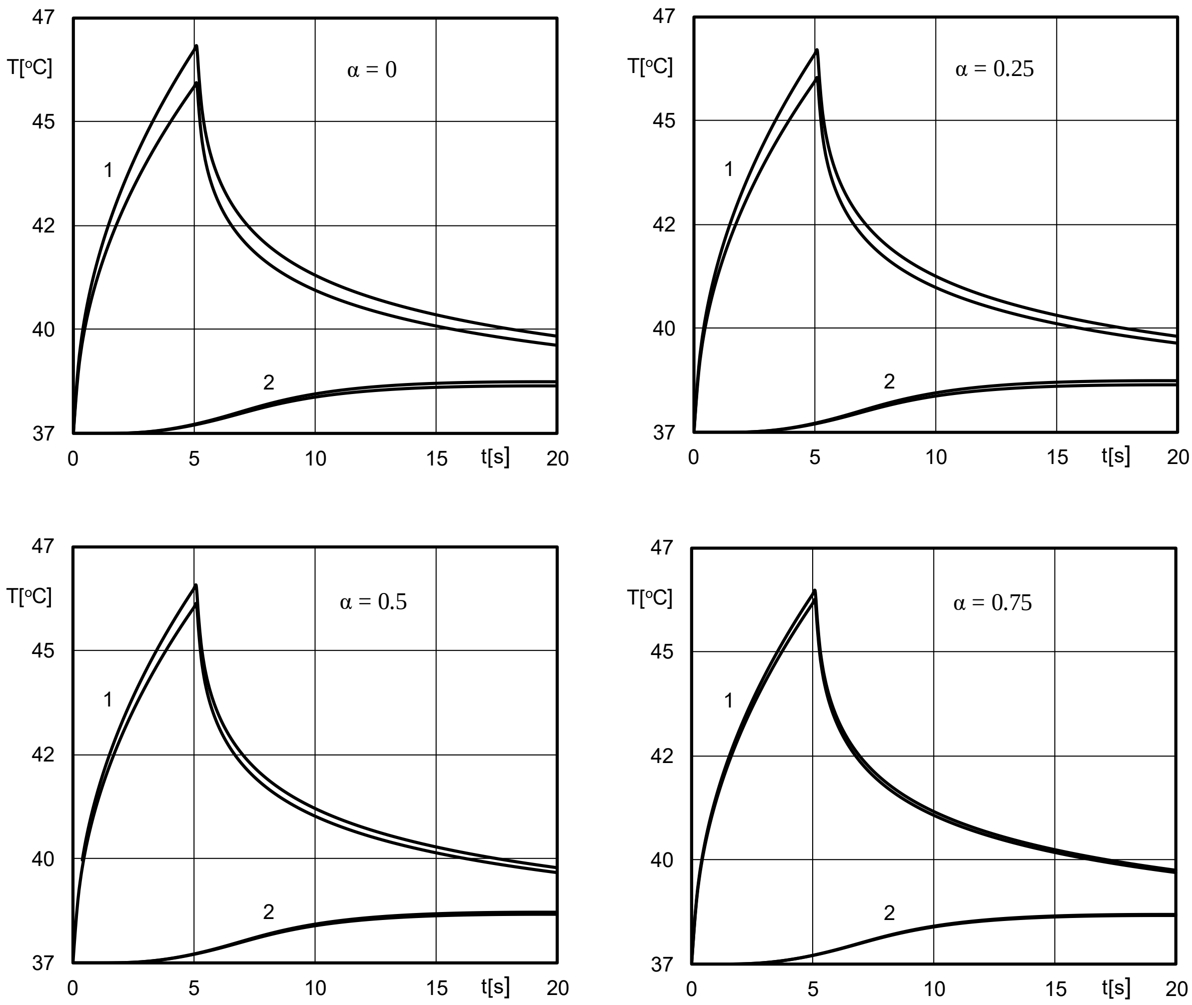

Figure 3: Thee heating and cooling curves for chosen values of parameter α

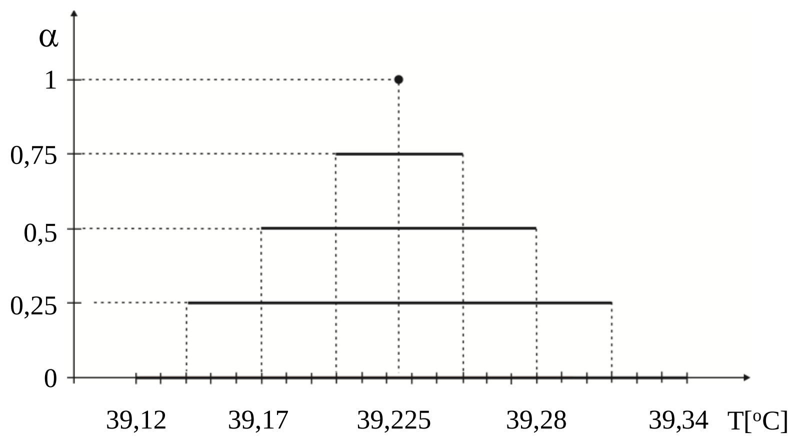

Figure 4: The interval temperature values at the node corresponding to the node L_1 after 20 s for chosen values of parameter α

6 CONCLUSIONS

In the paper, the numerical analysis of a heat transfer process proceeding in the non-homogeneous biological tissue domain has been presented. In the mathematical model, the thermophysical parameters of each skin's layer (epidermis, dermis, subcutaneous region) such as volumetric specific heat and thermal conductivity have been given as fuzzy numbers.

The problem analysed has been solved using a fuzzy version of the finite difference method (FDM) with α-cuts and the rules of directed interval arithmetic. Such an approach allows one to avoid complicated fuzzy arithmetic and treat the considered fuzzy numbers as interval numbers. For bigger values of α, the temperature interval is narrower. For $\alpha = 1$, the wideness of the temperature interval is equal to 0.

The fuzzy version of FDM allows one to find the numerical solution in the fuzzy form and such information may be important especially for the parameters which are difficult to estimate, for example tissue parameters.

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